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Parent actions, dualities and new Weyl-invariant actions of bosonic p-branes

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Abstract: By using the systematic approach of parent action method, we derive one Weyl-noninvariant and two Weyl-invariant actions of bosonic p-branes ($p \geq 2$) starting from the Nambu-Goto action, and establish the duality symmetries in this set of four actions. Moreover, we discover a new bosonic p-brane action (including the string theory) and deduce two corresponding Weyl-invariant formulations by proposing a new special parent action. We find that the same duality symmetries as those mentioned above exist in this new set of actions. The new p-brane actions are also briefly analyzed.

Keywords: p-branes, Conformal and W Symmetry.

Contents

1. Introduction

The parent or master action approach was proposed by Deser and Jackiw [1] nearly two decades ago. The purpose was to establish, at the level of the lagrangian instead of equations of motion, the equivalence or so-called duality between the abelian self-dual and Maxwell-Chern-Simons models in (2+1)-dimensional spacetime. The approach was in fact originated from the Legendre transformation and recently has been applied and developed in diverse directions. For instance, one direct development [2] is that it has been generalized to establish the duality between the non-abelian self-dual and Yang-Mills-Chern-Simons models. Another interesting application [3] is that the self-duality of various chiral p-form actions has been established with the modification [4] of the approach with one more auxiliary field to preserve the manifest Lorentz invariance of chiral p-forms.

The main idea of the parent action approach [5] contains two steps: (1) to introduce auxiliary fields and then construct a parent or master action based on a known action, and (2) to make variation of the parent action with respect to each auxiliary field, solve one auxiliary field in terms of other fields and then substitute the solution into the parent action. Through making variations with respect to different auxiliary fields, we can obtain different forms of the actions. The actions are, of course, equivalent classically, and the relation between them is usually referred to as duality. If the resulting actions are the same, the relation is called self-duality.

The bosonic p-branes [6] are extended objects that are embedded in a higher dimensional spacetime, and their dynamics is governed in general by the following four different kinds of actions. The first is the Nambu-Goto action (S_{NG}) that is proportional to the $(p+1)$-dimensional worldvolume by Nambu [7] and Goto [8] for a string ($p = 1$). The second action is the one with an auxiliary worldvolume metric and a cosmological term (S_P, where the subscript means p-branes), whose formulation of a membrane ($p=2$) was proposed by Howe and Tucker [9] in the construction of a locally supersymmetric invariant model of a spinning membrane. The two actions, S_{NG} and S_P, are classically equivalent to each other. Instead of the nonpolynomial form of the former, the latter is quadratic in the derivatives of spacetime coordinates at the price of introducing the auxiliary worldvolume metric. Quite noticeable is that S_P is not Weyl-invariant for the general cases, $p \geq 2$, but Weyl-invariant for the string theory, i.e., $p=1$. This string action was first given by Brink, Di Vecchia and Howe and Deser and Zumino [10, 11]. The other two p-brane actions possess a Weyl symmetry as demonstrated in the following. Because of the important role played by the Weyl symmetry in the covariant quantization of strings, the construction of p-brane actions with the Weyl invariance was then motivated. One recent proposal was to introduce an auxiliary scalar field in both the worldvolume and spacetime in the Weyl-noninvariant formulation S_P, and to propose a Weyl-invariant action of p-branes (S_W^{I}, called the first Weyl-invariant action in our paper) and a Weyl-invariant action of Dp-branes as well [12, 13, 14]. Of course, this Weyl-invariant p-brane action is classically equivalent to that constructed long time ago [15, 6] by the simple expedient of raising the action of Brink et al. for the string theory to the power $(p+1)/2$. The old Weyl-invariant action (S_W^{II}, called the second Weyl-invariant action) is nonpolynomial as the Nambu-Goto formulation, but contains no other auxiliary fields besides the auxiliary worldvolume metric.

As we know, the relation revealed at present among the four different p-brane actions is merely classical equivalence,[1] that is, the same equations of motion as those from the original action S_{NG} can be derived from S_P, S_W^{I}, and S_W^{II}. We may classify the four actions by polynomiality and by the Weyl symmetry. For the polynomiality, S_P and S_W^{I} belong to the class with quadratic derivatives of spacetime coordinates while S_{NG} and S_W^{II} constitute the class with nonpolynomiality; for the Weyl symmetry[2] S_P belongs to the class without the Weyl invariance while S_W^{I} and S_W^{II} constitute the class with such an invariance.

Because p-branes (including strings) can be described by different kinds of actions it has been a noticeable topic to derive from one formulation (usually the Nambu-Goto) the others systematically and to establish relations among them, such as duality. To this end a first-order action method was proposed [16], and the string action of Brink et al. with the Weyl invariance was derived from the Nambu-Goto action of strings[3] and the duality between the two formulations was established. The main idea of the method coincides with that of the parent action approach mentioned above, in both of which it is inevitable to introduce Lagrange multipliers and other auxiliary fields and then to make variations with

[1] The duality was discussed [16] in the context of first- and second-order actions.

[2] We do not consider the Weyl invariance of the Nambu-Goto action because it does not contain the worldvolume metric.

[3] For p-brane ($p \geq 2$) cases a Weyl-invariant action was constructed [16] from the Nambu-Goto through directly replacing the induced metric by a Weyl-invariant combination of the worldvolume metric and the induced metric itself. We do not include this action in our paper.

respect to them. This method was also applied to Dp-branes [16, 17]. Recently another attempt, i.e., a mechanism was suggested [13] to derive the Weyl-invariant actions of p-branes and Dp-branes. Here it should be pointed out that this mechanism is in fact an interesting application of the parent action approach as will be seen clearly in the following contents.

In this paper, we start with the Nambu-Goto action of bosonic p-branes and derive other p-brane actions by the systematic approach of parent action method. Simultaneously we also establish the duality symmetries among the four different p-brane actions. Incidentally, dual actions of Dp-branes were discussed in refs. [18, 19], but the discussions were limited to duality with respect to gauge fields. We note that this systematic method is powerful *not only* in the derivation of known actions *but also* in the discovery of new actions. In fact, by this method we propose a special parent action and find out a new p-brane action including the string theory. Furthermore, we obtain two corresponding Weyl-invariant actions and establish the same duality symmetries among the Nambu-Goto and three new actions as that among the four known actions.

Though the Nambu-Goto action of strings has geometrical meaning and is intuitively easy to understand, it is not very convenient for covariant quantization of the string. On the other hand, the string action of Brink et al. does not have geometrical meaning but is quite useful in the covariant quantization of strings. Thus it is useful to make various formulations of the actions. It remains to be seen if our new actions shed any insight in this direction.

The arrangement of this paper is as follows. In the next section, we first write down [13] the parent action of the Nambu-Goto formulation and derive S_P as the dual action of S_{NG}. Secondly we construct the parent action of S_P and obtain S_{NG} as the dual action of S_P. Thereby we establish the duality between S_{NG} and S_P with respect to the induced metric and worldvolume metric, respectively. In section 3, our discussions turn to the Weyl-invariant p-brane actions. Introducing an auxiliary scalar field Φ in both the worldvolume and spacetime and making the transformation $g_{ij} \to \Phi g_{ij}$, where g_{ij} is the worldvolume metric, in the parent action of the Nambu-Goto formulation, we can write down a different parent action [13]. The variation of this parent action leads to the first Weyl-invariant action as the dual action of the Nambu-Goto. Moreover, constructing the parent action of the first Weyl-invariant formulation, we derive the second Weyl-invariant action as the dual action of the first Weyl-invariant one. As a result, the Nambu-Goto and first Weyl-invariant actions are dual to each other with respect to the induced metric, and the first and second Weyl-invariant actions are dual to each other with respect to the scalar field. In section 4, we discuss the dual actions of the first and second Weyl-invariant formulations with respect to the worldvolume metric, and find that both of them are dual to S_{NG}, but not to S_P as is naively thought. In section 5, a special parent action of the Nambu-Goto formulation is proposed by defining the inverse of the induced metric. Completely following the procedure described in the preceding three sections, we deduce a new action that is dual to the Nambu-Goto and its two corresponding Weyl-invariant ones, and establish the same duality symmetries among the Nambu-Goto and three new actions as those among the four known actions. Finally a conclusion is made in section 6.

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The notation we use throughout this paper is as follows:

$$\eta_{\mu\nu} = \mathrm{diag}(-1, 1, \ldots, 1), \tag{1.1}$$

stands for the flat metric of the D-dimensional Minkowski spacetime. Greek indices $(\mu, \nu, \sigma, \ldots)$ run over $0, 1, \ldots, D-1$.

$$h_{ij} \equiv \frac{\partial X^\mu}{\partial \xi^i} \frac{\partial X^\nu}{\partial \xi^j} \eta_{\mu\nu}, \tag{1.2}$$

is defined as the induced metric in the $(p+1)$-dimensional worldvolume spanned by $p+1$ arbitrary parameters ξ^i. Latin indices $(i, j, k, \ldots)$ take the values $0, 1, \ldots, p$. Coordinates $X^\mu(\xi^i)$ are the D-component Lorentz vector field in the spacetime and D scalar fields in the worldvolume as well. Finally g_{ij} means the auxiliary worldvolume metric and g^{ij} its inverse.

2. Duality between S_{NG} and S_P

We begin with the Nambu-Goto action of bosonic p-branes

$$S_{NG} = -T_p \int d^{p+1}\xi \sqrt{-h}\,, \tag{2.1}$$

where $h \equiv \det(h_{ij})$ and T_p is the p-brane tension. According to the parent action approach [1, 5], we introduce the worldvolume second-rank tensor fields Λ^{ij} and g_{ij}, and write down the parent action of the Nambu-Goto formulation

$$S^{\mathrm{I}}_{\mathrm{parent}} = -T_p \int d^{p+1}\xi \left[\sqrt{-g} + \Lambda^{ij}\left(g_{ij} - h_{ij}\right)\right], \tag{2.2}$$

where $g \equiv \det(g_{ij})$. We will see later that g_{ij} is just the auxiliary worldvolume metric, but at present it is treated as an independent auxiliary field. Note that eq. (2.2)[4] was obtained in ref. [13] by the consideration that coincides with the parent action approach.

Now varying eq. (2.2) with respect to Λ^{ij} gives the relation $g_{ij} = h_{ij}$, together with which eq. (2.2) turns back to the Nambu-Goto action (2.1). This shows the classical equivalence between the parent and Nambu-Goto actions. However, varying eq. (2.2) with respect to g_{ij} leads to the expression of Λ^{ij} in terms of g_{ij}:

$$\Lambda^{ij} = -\frac{1}{2}\sqrt{-g}g^{ij}\,, \tag{2.3}$$

where g^{ij} is the inverse of g_{ij}. Substituting eq. (2.3) into eq. (2.2), we obtain the dual version of the Nambu-Goto action

$$S_P = -\frac{T_p}{2} \int d^{p+1}\xi \sqrt{-g}\left[g^{ij}h_{ij} - (p-1)\right]. \tag{2.4}$$

This is the p-brane action with the auxiliary field g_{ij} that now plays the role of the worldvolume metric and with the cosmological term for $p \geq 2$. For the string theory ($p = 1$),

[4] The string case of this equation appeared in ref. [16].

it reduces to the Brink-Di Vecchia-Howe-Deser-Zumino action as we noted in the previous section. In the above, we have shown that S_P is dual to S_{NG} with respect to the induced metric h_{ij}. We note, however, that eq. (2.4) is not Weyl-invariant for $p \geq 2$.

Parent actions are not unique. Starting from S_P, we construct the parent action

$$S^{\rm II}_{\rm parent} = -\frac{T_p}{2}\int d^{p+1}\xi \left[\sqrt{-G}\left(G^{ij}h_{ij} - (p-1)\right) + \Lambda_{ij}\left(G^{ij} - g^{ij}\right)\right], \tag{2.5}$$

where Λ_{ij} and G^{ij} are two auxiliary second-rank tensor fields introduced in the worldvolume, $G \equiv \det(G_{ij})$ and G_{ij} is the inverse of G^{ij}, and vice versa. We can verify that S_{NG} is dual to S_P with respect to the inverse of the worldvolume metric g^{ij}. The procedure of the verification is as follows. First making the variation of eq. (2.5) with respect to Λ_{ij} gives $G^{ij} = g^{ij}$, with which eq. (2.5) reduces to the p-brane action (2.4). This means that the second parent action is classically equivalent to S_P. Secondly, to make the variation of eq. (2.5) with respect to G^{ij} leads to the relation of Λ_{ij} expressed by h_{ij} and G_{ij}:

$$\Lambda_{ij} = \frac{1}{2}\sqrt{-G}G_{ij}\left(G^{kl}h_{kl} - (p-1)\right) - \sqrt{-G}h_{ij}\,, \tag{2.6}$$

which is independent of g^{ij}. After substituting eq. (2.6) into eq. (2.5) and making the variation again with respect to g^{ij} which is now dealt with as a Lagrange multiplier, we simply derive $\Lambda_{ij} = 0$, that is,

$$\frac{1}{2}G_{ij}\left(G^{kl}h_{kl} - (p-1)\right) - h_{ij} = 0\,. \tag{2.7}$$

The solution for the $p \geq 2$ cases is

$$G_{ij} = h_{ij}\,. \tag{2.8}$$

Using eqs. (2.6)–(2.8), we then obtain from eq. (2.5) the Nambu-Goto action. As for the string case, i.e., $p = 1$, eq. (2.7) is simplified to be

$$\frac{1}{2}G_{ij}G^{kl}h_{kl} - h_{ij} = 0\,, \tag{2.9}$$

whose solution is

$$G_{ij} = f_1\left(\xi^0, \xi^1\right)h_{ij}\,, \tag{2.10}$$

where $f_1(\xi^0, \xi^1)$ is an arbitrary scalar function of the worldsheet parameters. This arbitrary function appears due to the Weyl invariance existing in the action of Brink et al. for the string theory. By using eqs. (2.6), (2.9) and (2.10) and the condition $p = 1$, we have from eq. (2.5) the Nambu-Goto action for strings as expected. Hence the dual action deduced from the second parent action is exactly the Nambu-Goto action as we have claimed.

The duality established above may be illustrated as in figures 1 and 2 as is done in ref. [5].

$$\begin{tikzcd}
 & S^{\mathrm{I}}_{\mathrm{parent}} \arrow[dl, "\delta\Lambda^{ij}"'] \arrow[dr, "\delta g_{ij}"] & \\
S_{NG} \arrow[rr, leftrightarrow, "\textbf{dual}"'] & & S_P
\end{tikzcd}$$

Figure 1: Schematic relation of the actions: The parent action $S^{\mathrm{I}}_{\mathrm{parent}}$ is classically equivalent to S_{NG} and S_P, and S_{NG} and S_P are dual to each other with respect to the induced metric.

$$\begin{tikzcd}
 & S^{\mathrm{II}}_{\mathrm{parent}} \arrow[dl, "\delta\Lambda_{ij}"'] \arrow[dr, "\delta G^{ij} + \delta g^{ij}"] & \\
S_P \arrow[rr, leftrightarrow, "\textbf{dual}"'] & & S_{NG}
\end{tikzcd}$$

Figure 2: Schematic relation of the actions: The parent action $S^{\mathrm{II}}_{\mathrm{parent}}$ shows that S_P and S_{NG} are dual to each other with respect to the worldvolume metric.

3. Dualities between S_{NG} and S^{I}_W and between S^{I}_W and S^{II}_W

Introducing an auxiliary scalar field $\Phi(\xi^i)$, and rescaling the worldvolume metric $g_{ij} \to \Phi g_{ij}$ in the parent action of the Nambu-Goto (2.2), we write down the third parent action [13]

$$S^{\mathrm{III}}_{\mathrm{parent}} = -T_p \int d^{p+1}\xi \left[\Phi^{(p+1)/2}\sqrt{-g} + \Lambda^{ij}\left(\Phi g_{ij} - h_{ij}\right)\right], \tag{3.1}$$

where $\Phi(\xi^i)$ should be a scalar in both the spacetime and worldvolume in order to keep eq. (3.1) invariant under the Lorentz transformation and reparametrization.

Varying eq. (3.1) with respect to Λ^{ij} brings about $g_{ij} = \Phi^{-1}h_{ij}$, which leads to nothing new but the classical equivalence between the Nambu-Goto and third parent actions. However, varying the equation with respect to g_{ij}, we get Λ^{ij} as

$$\Lambda^{ij} = -\frac{1}{2}\Phi^{(p-1)/2}\sqrt{-g}g^{ij}\,. \tag{3.2}$$

Substituting eq. (3.2) back to eq. (3.1), we derive the dual action

$$S^{\mathrm{I}}_W = -\frac{T_p}{2}\int d^{p+1}\xi\sqrt{-g}\left[\Phi^{(p-1)/2}g^{ij}h_{ij} - (p-1)\Phi^{(p+1)/2}\right], \tag{3.3}$$

which was obtained in ref. [13] but whose duality was not uncovered. This action is interesting because it is invariant under the Weyl transformation

$$\begin{aligned}
X^\mu(\xi) &\longrightarrow X^\mu(\xi)\,,\\
g_{ij}(\xi) &\longrightarrow \exp\left(\omega(\xi)\right) g_{ij}(\xi)\,,\\
\Phi(\xi) &\longrightarrow \exp\left(-\omega(\xi)\right)\Phi(\xi)\,,
\end{aligned} \tag{3.4}$$

where $\omega(\xi)$ is an arbitrary real function of worldvolume parameters. We thus call it the first Weyl-invariant p-brane action in this paper. For the string theory, i.e., $p = 1$, S^{I}_W turns back to the Weyl-invariant action of Brink et al. The duality between S_{NG} and S^{I}_W may be illustrated as in figure 3.

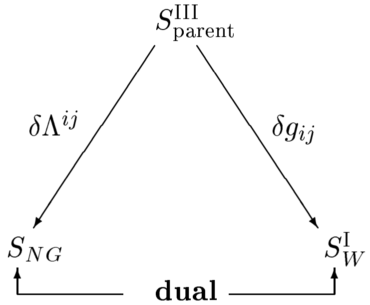

Figure 3: Schematic relation of the actions: The parent action $S^{\mathrm{III}}_{\mathrm{parent}}$ shows that S_{NG} and S^{I}_{W} are dual to each other with respect to the induced metric.

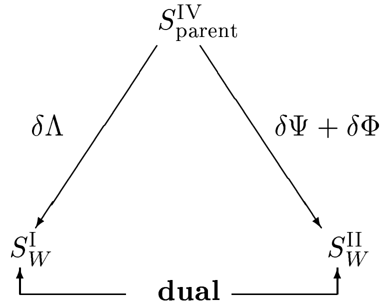

Figure 4: Schematic relation of the actions: The parent action $S^{\mathrm{IV}}_{\mathrm{parent}}$ shows that S^{I}_{W} and S^{II}_{W} are dual to each other with respect to the scalar field.

Next we introduce two auxiliary scalar fields Λ and Ψ, and further construct the fourth parent action which corresponds to S^{I}_{W}

$$S^{\mathrm{IV}}_{\mathrm{parent}} = -\frac{T_p}{2}\int d^{p+1}\xi \left[\sqrt{-g}\left(\Psi^{(p-1)/2} g^{ij} h_{ij} - (p-1)\Psi^{(p+1)/2}\right) + \Lambda\left(\Psi - \Phi\right)\right]. \quad (3.5)$$

It is trivial to make the variation of eq. (3.5) with respect to Λ, which just leads to the classical equivalence between $S^{\mathrm{IV}}_{\mathrm{parent}}$ and S^{I}_{W}. However, it is nontrivial to vary eq. (3.5) with respect to Ψ, from which we deduce the relation

$$\Lambda = -\frac{p-1}{2}\Psi^{(p-3)/2}\sqrt{-g}\left(g^{ij}h_{ij} - (p+1)\Psi\right), \quad (3.6)$$

which is independent of Φ. Substituting eq. (3.6) into eq. (3.5) and varying the parent action again with respect to Φ treated as a Lagrange multiplier now, we obtain $\Lambda = 0$, that is,

$$\Psi = \frac{1}{p+1} g^{ij} h_{ij}\,, \quad (3.7)$$

for the $p \geq 2$ cases. Using eqs. (3.6) and (3.7), we then derive from $S^{\mathrm{IV}}_{\mathrm{parent}}$ the dual action

$$S^{\mathrm{II}}_{W} = -T_p \int d^{p+1}\xi \sqrt{-g}\left(\frac{1}{p+1} g^{ij} h_{ij}\right)^{(p+1)/2}. \quad (3.8)$$

This p-brane action [15, 6] also preserves invariance under the Weyl transformation (3.4), and is called here the second Weyl-invariant action that does not contain auxiliary scalar fields. For $p = 1$, eq. (3.5) simply reduces to the Weyl-invariant action of Brink et al. for the string theory. Therefore, the duality between S^{I}_{W} and S^{II}_{W} is established and may be illustrated as in figure 4.

4. Dualities between S^{I}_{W} and S_{NG} and between S^{II}_{W} and S_{NG}

In order to acquire a whole understanding of the dualities among S_{NG}, S_P, S^{I}_{W}, and S^{II}_{W}, we have to exhaust all parent actions. To this end, we construct the remaining parent actions

$$S^{\rm V}_{\rm parent} = -\frac{T_p}{2}\int d^{p+1}\xi \left[\sqrt{-G}\left(\Phi^{(p-1)/2}G^{ij}h_{ij} - (p-1)\Phi^{(p+1)/2}\right) + \Lambda_{ij}\left(G^{ij} - g^{ij}\right)\right], \tag{4.1}$$

and

$$S^{\rm VI}_{\rm parent} = -T_p\int d^{p+1}\xi \left[\sqrt{-G}\left(\frac{1}{p+1}G^{ij}h_{ij}\right)^{(p+1)/2} + \Lambda_{ij}\left(G^{ij} - g^{ij}\right)\right]. \tag{4.2}$$

They correspond to $S^{\rm I}_W$ and $S^{\rm II}_W$, respectively, which can be easily seen by varying them with respect to Λ_{ij}. On the other hand, varying with respect to G^{ij}, we calculate Λ_{ij} from eq. (4.1):

$$\Lambda_{ij} = \frac{1}{2}\sqrt{-G}G_{ij}\left(\Phi^{(p-1)/2}G^{kl}h_{kl} - (p-1)\Phi^{(p+1)/2}\right) - \sqrt{-G}\Phi^{(p-1)/2}h_{ij}, \tag{4.3}$$

and from eq. (4.2):

$$\Lambda_{ij} = \frac{1}{2}\sqrt{-G}\left(\frac{1}{p+1}G_{ij}G^{kl}h_{kl} - h_{ij}\right)\left(\frac{1}{p+1}G^{mn}h_{mn}\right)^{(p-1)/2}. \tag{4.4}$$

It is important to note that both eq. (4.3) and eq. (4.4) are independent of g^{ij}. Upon substituting eq. (4.3) and eq. (4.4) into $S^{\rm V}_{\rm parent}$ and $S^{\rm VI}_{\rm parent}$, respectively, and varying the parent actions again with respect to g^{ij} dealt with as Lagrange multipliers, we simply have $\Lambda_{ij} = 0$ because of the independence of the worldvolume metric. Therefore, eqs. (4.3) and (4.4) are simplified to be

$$\frac{1}{2}G_{ij}\left(G^{kl}h_{kl} - (p-1)\Phi\right) - h_{ij} = 0\,, \tag{4.5}$$

and

$$\frac{1}{p+1}G_{ij}G^{kl}h_{kl} - h_{ij} = 0\,, \tag{4.6}$$

respectively. Now let us solve eqs. (4.5) and (4.6). Multiplying G^{ij} on both sides of eq. (4.5) and considering $p \geq 2$, we have

$$\Phi = \frac{1}{p+1}G^{ij}h_{ij}\,. \tag{4.7}$$

By using this relation, we find that eq. (4.5) reduces to eq. (4.6). Thus their solutions are

$$G_{ij} = f_2(\xi)h_{ij}\,, \tag{4.8}$$

where $f_2(\xi)$ is an arbitrary scalar function of the worldvolume parameters ξ^i, and it exists because $S^{\rm I}_W$ and $S^{\rm II}_W$ have the Weyl invariance as explained before. Substituting eqs. (4.3), (4.5), (4.7) and (4.8) into $S^{\rm V}_{\rm parent}$, and eqs. (4.4), (4.6) and (4.8) into $S^{\rm VI}_{\rm parent}$, we arrive at the same dual action, the Nambu-Goto action S_{NG}. As a result, both $S^{\rm I}_W$ and $S^{\rm II}_W$ are dual to S_{NG} with respect to the worldvolume metric, and their dualities may be illustrated as in figures 5 and 6.

Starting from the Nambu-Goto action, we have derived the other three p-brane actions S_P, $S^{\rm I}_W$, and $S^{\rm II}_W$ by using our systematic parent action approach [1, 5]. Simultaneously,

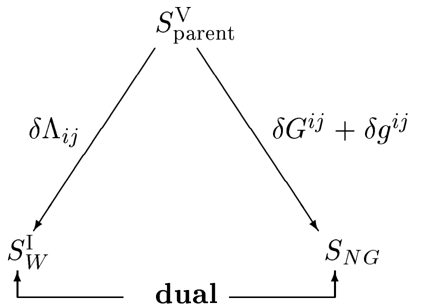

Figure 5: Schematic relation of the actions: The parent action $S^{\mathrm{V}}_{\text{parent}}$ shows that S^{I}_{W} and S_{NG} are dual to each other with respect to the worldvolume metric.

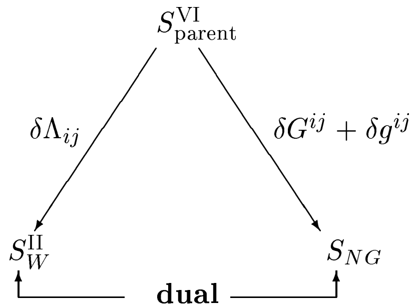

Figure 6: Schematic relation of the actions: The parent action $S^{\mathrm{VI}}_{\text{parent}}$ shows that S^{II}_{W} and S_{NG} are dual to each other with respect to the worldvolume metric.

we have found all the parent actions that can be constructed from the four actions, and have established their duality symmetries as shown in figures 1–6. It is quite interesting that S_{NG} and S_P, though dual to each other, are not in an equal position even at the classical level among the four dual actions, because S^{I}_W and S^{II}_W are related to S_{NG} with *direct* dualities (see figures 3, 5 and 6) but not to S_P with such dualities. Thus the natural higher dimensional generalization of the action of Brink et al. for the strings is the Weyl-invariant action (3.8) not only because it is directly related to the Nambu-Goto action but also because the string action of Brink et al. is Weyl invariant in contrast to the naive generalization (2.4).

We may precisely describe the well-known classical equivalence between S_{NG} and S_P "in the sense of equations of motion". On the other hand, we also notice the non-equality between S_{NG} and S_P from the point of view of the number of lagrangians. If we set Φ to a constant a in S^{I}_W in eq. (3.3), we obtain *an infinite number* (due to the arbitrary constant a) of so-called S_P-type actions [18]

$$S'_P = -\frac{T'_p}{2} \int d^{p+1}\xi \sqrt{-g} \left[g^{ij} h_{ij} - (p-1)a \right], \tag{4.9}$$

where $T'_p = a^{(p-1)/2} T_p$, and the $a = 1$ case corresponds to S_P. They are classically equivalent to the *unique* S_{NG} after using the equations of motion.

5. New bosonic p-brane actions and dualities

In section 1, we mentioned that the parent action approach can also help us to find new p-brane actions that are different from the four known ones. Now we turn to the discussion in this aspect.

Let us begin with the Nambu-Goto action as the starting point of our systematic method, and construct a new parent action as follows:

$$A^{\mathrm{I}}_{\text{parent}} = -T_p \int d^{p+1}\xi \left[\sqrt{-g} + \Lambda_{ij} \left(g^{ij} - h^{ij} \right) \right]. \tag{5.1}$$

At first sight it looks like $S^{\rm I}_{\rm parent}$ in eq. (2.2), just with the exchange of lower and upper indices in the second term. We note that if h^{ij} is understood as $g^{ik}g^{jl}h_{kl}$, $A^{\rm I}_{\rm parent}$ is exactly the same as $S^{\rm I}_{\rm parent}$ and nothing new can be deduced from eq. (5.1) but the p-brane action (2.4). Actually h^{ij} in eq. (5.1) is defined as the inverse of h_{ij} and is independent of g_{ij}. With this in mind, we follow the usual procedure described in section 2 and derive a new p-brane action that is dual and equivalent to S_{NG}:

$$A_P = -\frac{T_p}{2}\int d^{p+1}\xi\sqrt{-g}\left[-g_{ij}h^{ij} + (p+3)\right]. \tag{5.2}$$

For the string theory ($p=1$), we can prove

$$g_{ij}h^{ij} = gh^{-1}g^{ij}h_{ij}\,, \qquad (i,j=0,1)\,, \tag{5.3}$$

and then obtain a new action for strings

$$A_P(\text{string}) = -\frac{T_1}{2}\int d^2\xi\sqrt{-g}\left(-gh^{-1}g^{ij}h_{ij}+4\right), \qquad (i,j=0,1)\,. \tag{5.4}$$

The new p-brane action is Lorentz invariant and reparametrization invariant as well. Under the reparametrization, the factor $d^{p+1}\xi\sqrt{-g}$ remains unchanged and g_{ij} and h^{ij} transform as the worldvolume covariant and contravariant tensors, respectively,

$$\begin{aligned} g_{ij}(\xi) &= \frac{\partial\xi'^k}{\partial\xi^i}\frac{\partial\xi'^l}{\partial\xi^j}g'_{kl}(\xi')\,,\\ h^{ij}(\xi) &= \frac{\partial\xi^i}{\partial\xi'^k}\frac{\partial\xi^j}{\partial\xi'^l}h'^{kl}(\xi')\,, \end{aligned} \tag{5.5}$$

which keeps $g_{ij}h^{ij}$ invariant. Unfortunately, the action (5.2) does not possess the Weyl invariance under (3.4) *even* for the string theory. However, this invariance can be recovered by introducing an auxiliary scalar field by the procedure done for S_P in section 3, which will be shown below. On the other hand, A_P and $A_P(\text{string})$ are nonpolynomial in the derivatives of spacetime coordinates because of our special definition of h^{ij}. But this brings in nothing strange as S_{NG} possesses such a nonpolynomiality intrinsically.

For the new p-brane action, we can write down its corresponding parent action

$$A^{\rm II}_{\rm parent} = -\frac{T_p}{2}\int d^{p+1}\xi\left[\sqrt{-G}\left(-G_{ij}h^{ij}+(p+3)\right)+\Lambda^{ij}\left(G_{ij}-g_{ij}\right)\right]. \tag{5.6}$$

Following the similar treatment as done in section 2 we arrive at S_{NG} as the dual action. The duality between S_{NG} and A_P is thus established and illustrated in figures 7 and 8.

Now let us restore the Weyl invariance of the new p-brane action. As was done in section 3, introducing an auxiliary scalar field $\Phi(\xi^i)$, and rescaling the worldvolume metric as Φg_{ij} in $A^{\rm I}_{\rm parent}$ in eq. (5.1), we construct a new parent action

$$A^{\rm III}_{\rm parent} = -T_p\int d^{p+1}\xi\left[\Phi^{(p+1)/2}\sqrt{-g}+\Lambda_{ij}\left(\Phi^{-1}g^{ij}-h^{ij}\right)\right]. \tag{5.7}$$

$$\begin{tikzcd} & A^{\mathrm{I}}_{\text{parent}} \arrow[dl, "\delta\Lambda_{ij}"'] \arrow[dr, "\delta g^{ij}"] & \\ S_{NG} \arrow[rr, leftrightarrow, "\textbf{dual}"'] & & A_P \end{tikzcd}$$

Figure 7: Schematic relation of the actions: The parent action $A^{\mathrm{I}}_{\text{parent}}$ shows that S_{NG} and A_P are dual to each other with respect to the inverse of the induced metric.

$$\begin{tikzcd} & A^{\mathrm{II}}_{\text{parent}} \arrow[dl, "\delta\Lambda^{ij}"'] \arrow[dr, "\delta G_{ij} + \delta g_{ij}"] & \\ A_P \arrow[rr, leftrightarrow, "\textbf{dual}"'] & & S_{NG} \end{tikzcd}$$

Figure 8: Schematic relation of the actions: The parent action $A^{\mathrm{II}}_{\text{parent}}$ shows that A_P and S_{NG} are dual to each other with respect to the worldvolume metric.

According to the steps of the parent action approach described in detail previously, we deduce the dual action

$$A^{\mathrm{I}}_W = -\frac{T_p}{2}\int d^{p+1}\xi\sqrt{-g}\left[-\Phi^{(p+3)/2}g_{ij}h^{ij} + (p+3)\Phi^{(p+1)/2}\right]. \tag{5.8}$$

It is obviously invariant under the Weyl transformation (3.4), and is referred to as the first new Weyl-invariant action. As to the string case, A^{I}_W simplifies to

$$A^{\mathrm{I}}_W(\text{string}) = -\frac{T_1}{2}\int d^2\xi\sqrt{-g}\left(-\Phi^2 gh^{-1}g^{ij}h_{ij} + 4\Phi\right), \qquad (i,j=0,1). \tag{5.9}$$

It is possible to make its hamiltonian analysis by the method in ref. [20] that is suitable to the nonpolynomial S^{II}_W in eq. (3.8).

Next starting from A^{I}_W, we construct a further parent action

$$A^{\mathrm{IV}}_{\text{parent}} = -\frac{T_p}{2}\int d^{p+1}\xi\left[\sqrt{-g}\left(-\Psi^{(p+3)/2}g_{ij}h^{ij} + (p+3)\Psi^{(p+1)/2}\right) + \Lambda\left(\Psi - \Phi\right)\right]. \tag{5.10}$$

By following the same procedure utilized to derive S^{II}_W, we obtain the action dual to A^{I}_W:

$$A^{\mathrm{II}}_W = -T_p\int d^{p+1}\xi\sqrt{-g}\left(\frac{1}{p+1}g_{ij}h^{ij}\right)^{-(p+1)/2}, \tag{5.11}$$

which is evidently invariant under the Weyl transformation (3.4), and is named the second new Weyl-invariant action. For the string theory, A^{II}_W reduces to

$$A^{\mathrm{II}}_W(\text{string}) = 2T_1\int d^2\xi\frac{h}{\sqrt{-g}g^{ij}h_{ij}}, \qquad (i,j=0,1). \tag{5.12}$$

Note that $A^{\mathrm{I}}_W(\text{string})$ and $A^{\mathrm{II}}_W(\text{string})$ have different forms but, of course, dual and equivalent, while both S^{I}_W and S^{II}_W lead to the string action of Brink et al. The dualities between S_{NG} and A^{I}_W and between A^{I}_W and A^{II}_W are shown in figures 9 and 10, respectively.

The final task of this section is to write down the remaining parent actions that can be constructed from the action set, (S_{NG}, A_P, A^{I}_W, A^{II}_W):

$$A^{\mathrm{V}}_{\text{parent}} = -\frac{T_p}{2}\int d^{p+1}\xi\left[\sqrt{-G}\left(-\Phi^{(p+3)/2}G_{ij}h^{ij} + (p+3)\Phi^{(p+1)/2}\right) + \Lambda^{ij}\left(G_{ij} - g_{ij}\right)\right], \tag{5.13}$$

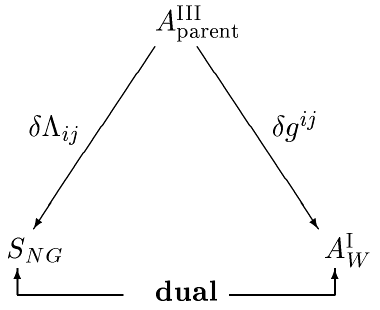

Figure 9: Schematic relation of the actions: The parent action $A^{\rm III}_{\rm parent}$ shows that S_{NG} and $A^{\rm I}_W$ are dual to each other with respect to the inverse of the induced metric.

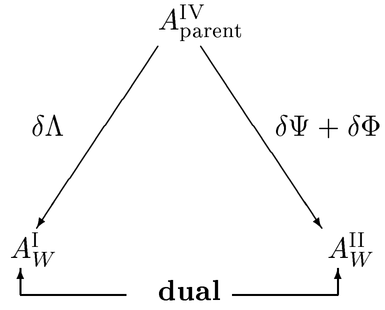

Figure 10: Schematic relation of the actions: The parent action $A^{\rm IV}_{\rm parent}$ shows that $A^{\rm I}_W$ and $A^{\rm II}_W$ are dual to each other with respect to the scalar field.

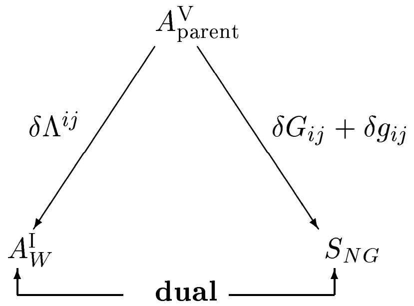

Figure 11: Schematic relation of the actions: The parent action $A^{\rm V}_{\rm parent}$ shows that $A^{\rm I}_W$ and S_{NG} are dual to each other with respect to the worldvolume metric.

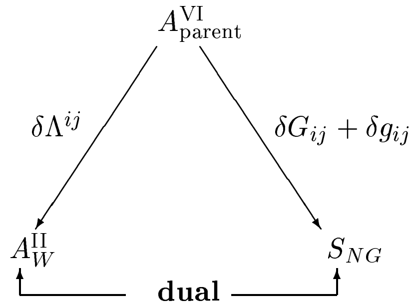

Figure 12: Schematic relation of the actions: The parent action $A^{\rm VI}_{\rm parent}$ shows that $A^{\rm II}_W$ and S_{NG} are dual to each other with respect to the worldvolume metric.

and

$$A^{\rm VI}_{\rm parent} = -T_p \int d^{p+1}\xi \left[\sqrt{-G}\left(\frac{1}{p+1}G_{ij}h^{ij}\right)^{-(p+1)/2} + \Lambda^{ij}\left(G_{ij} - g_{ij}\right)\right]. \qquad (5.14)$$

Then we can calculate the corresponding dual actions. The calculation procedure is nothing different from that described in section 4. So we omit the details and just state the result that both $A^{\rm I}_W$ and $A^{\rm II}_W$ are dual to S_{NG}, but not to A_P, which shows a kind of non-equality of S_{NG} and A_P in the set of dual actions, $(S_{NG}, A_P, A^{\rm I}_W, A^{\rm II}_W)$. This non-equality is similar to that of S_{NG} and S_P described in section 4. Furthermore, if we set Φ to a constant b in $A^{\rm I}_W$ in eq. (5.8), we obtain an infinite number of A_P-type actions

$$A'_P = -\frac{T''_p}{2}\int d^{p+1}\xi\sqrt{-g}\left[-g_{ij}h^{ij} + (p+3)b^{-1}\right], \qquad (5.15)$$

where $T''_p = b^{(p+3)/2}T_p$, and the $b = 1$ case coincides with A_P. They are equivalent to the unique A_P upon using the equations of motion. The dualities between $A^{\rm I}_W$ and S_{NG} and between $A^{\rm II}_W$ and S_{NG} are illustrated in figures 11 and 12, respectively.

6. Conclusion

By using the parent action approach of duality analysis, we construct the parent actions that correspond to the four actions of bosonic p-branes ($p \geq 2$), S_{NG}, S_P, S_W^{I}, and S_W^{II}, and establish the duality symmetries among them: (i) S_{NG} and S_P are dual to each other (see figures 1 and 2), (ii) the first and second Weyl-invariant actions with and without an auxiliary scalar field, respectively, S_W^{I} and S_W^{II}, are also dual to each other (see figures 3 and 4), and (iii) the two Weyl-invariant actions are dual to S_{NG}, but not to S_P (see figures 5 and 6). This means that S_{NG} and S_P, though dual to each other, are not in an equal position even at the classical level in the set of the four dual actions, (S_{NG}, S_P, S_W^{I}, S_W^{II}). In fact, starting from the Weyl-invariant p-brane action with an auxiliary scalar field, S_W^{I}, we can deduce an infinite number of so-called S_P-type actions (4.9) that are classically equivalent to the unique Nambu-Goto action upon using the equations of motion. This also shows a kind of non-equality between S_{NG} and S_P from the point of view of the number of lagrangians.

Furthermore, by proposing a parent action with the special definition of the inverse of the induced metric, we discover a new bosonic p-brane action (including the string case) which is nonpolynomial in the derivatives of spacetime coordinates and equivalent to the Nambu-Goto action classically. We subsequently obtain two corresponding Weyl-invariant actions and establish the same duality symmetries as above (see figures 7–12) among the Nambu-Goto and three new actions, (S_{NG}, A_P, A_W^{I}, A_W^{II}). The similar non-equality exists between S_{NG} and A_P as that between S_{NG} and S_P. We may concisely illustrate the dualities established in the two sets of dual actions, (S_{NG}, S_P, S_W^{I}, S_W^{II}) and (S_{NG}, A_P, A_W^{I}, A_W^{II}) in figures 13 and 14, respectively.

Our result shows that the Nambu-Goto action is a kind of source from which we can systematically derive its six dual actions in terms of the parent action approach of duality analysis. In string theories, it was quite useful to have various formulations of the actions for understanding the geometrical meaning and covariant quantization. We hope that our systematic investigation of the various formulations may be useful for these purposes. Further studies of the new p-brane actions are now being considered, such as

$$\begin{tikzcd} S_{NG} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & S_P \\ S_W^{\mathrm{I}} \arrow[r, leftrightarrow] & S_W^{\mathrm{II}} \end{tikzcd}$$

Figure 13: Dualities in the set of dual actions, (S_{NG}, S_P, S_W^{I}, S_W^{II}). No *direct* dualities exist between S_W^{I} and S_P and between S_W^{II} and S_P.

$$\begin{tikzcd} S_{NG} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & A_P \\ A_W^{\mathrm{I}} \arrow[r, leftrightarrow] & A_W^{\mathrm{II}} \end{tikzcd}$$

Figure 14: Dualities in the set of dual actions, (S_{NG}, A_P, A_W^{I}, A_W^{II}). No *direct* dualities exist between A_W^{I} and A_P and between A_W^{II} and A_P.

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the hamiltonian analysis, the supersymmetric extension, etc. Finally we point out that our discussions can also be applied to Dp-branes and similar conclusions can be made.

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